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A zero-bounded model of operant demand

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Contemporary approaches for evaluating the demand for reinforcers use either the Exponential or the Exponentiated model of operant demand, both derived from the framework of Hursh and Silberberg (2008). This report summarizes the strengths and complications of this framework and proposes a novel implementation. This novel implementation incorporates earlier strengths and resolves existing shortcomings that are due to the use of a logarithmic scale for consumption. The Inverse Hyperbolic Sine (IHS) transformation is reviewed and evaluated as a replacement for the logarithmic scale in models of operant demand. Modeling consumption in the "$\log_{10}$-like" IHS scale reflects relative changes in consumption (as with a log scale) and accommodates a true zero bound (i.e., zero consumption values). The presence of a zero bound obviates the need for a separate span parameter (i.e., k) and the span of the model may be more simply defined by maximum demand at zero price (i.e., Q_0). Further, this reformulated model serves to decouple the exponential rate constant (i.e., α) from variations in span, thus normalizing the rate constant to the span of consumption in IHS units and permitting comparisons when spans vary. This model, called the Zero-bounded Exponential (ZBE), is evaluated using simulated and real-world data. The direct reinstatement ZBE model showed strong correspondence with empirical indicators of demand and with a normalization of α (ZBEn) across empirical data that varied in reinforcing efficacy (dose, time to onset of peak effects). Future directions in demand curve analysis are discussed with recommendations for additional replication and exploration of scales beyond the logarithm when accommodating zero consumption data.

Key words: operant demand, behavioral economics, decision making, quantitative modeling

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Behavioral economic methods are increasingly applied in various areas of basic and applied science. Among the methods in the behavioral economic framework, the concept of demand has been particularly useful in quantifying complex cost–reinforcer relationships, such as those observed in issues such as substance use (Acuff et al., 2020; Bickel et al., 2014; Kaplan, Foster, et al., 2018; Reed et al., 2020), obesity (Epstein et al., 2012), and the consumption of drugs (Aston et al., 2015; Pickover et al., 2016). Briefly, the behavioral economic concept of demand refers to the degree to which an individual or group will work to defend their bliss point consumption of a reinforcer as a function of one or more factors, for example, price (Bickel et al., 2017; MacKillop et al., 2019; Schwartz et al., 2019). Further, this framework has also been used to evaluate the abuse liability of novel drugs (Aston et al., 2017; Owens et al., 2015) as well as the environmental and genetic factors that influence their use and abuse liability (Aston et al., 2017; Owens et al., 2015).

Analyses of operant demand typically evaluate changes in consumption as a function of changes in price using the logarithm, or *log*, scale (Lea & Roper, 1977). The log scale reflects changes proportionally and this trait is well suited to applications in economics that focus on *relative* changes. For instance, the log scale is one way to facilitate comparisons of how proportional decreases in consumption relate to proportional increases in price, that is, elasticity (Gilroy et al., 2020). Stated another way, "the important property of the elasticity of a function is that it is a number which is independent of the units in which the variables are measured. This is clear since the elasticity is

Simulations were adapted and completed by SPG. Analyses completed in the R Statistical Program by SPG and BAK. Dose-dependent data were furnished by SRH and analyses in GraphPad were completed by LPS and SRH. The adaptation and extension of the Exponential Model was the product of collaboration across all authors.

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defined in terms of proportional changes which are necessarily independent of units" (Allen, 1938, p. 252).

The prototypical form of the demand curve in log–log dimensions is nonlinear in shape. Hence, a nonlinear model is well-suited to represent this form. The model proposed by Hursh and Silberberg (2008) is an exponential decay function on the logarithm of consumption. For the sake of brevity, we shall refer to this model as the Exponential model (EXPL). This model evaluates logarithmic changes in consumption (i.e., $\log_{10}Q$; relative differences) as a function of price and its structure is shown in Equation 1:

$$log_{10}Q = log_{10}Q_0 + k\left(e^{-\alpha Q_0 P} - 1\right) \qquad (1)$$

In Equation 1, Q refers to consumption, P to price or cost, and the three fitted parameters are Q_0, α, and k. The rate constant of the exponential, α, captures the rate of change in consumption across the full domain of the demand curve (i.e., the sensitivity to increasing price). The y-intercept, Q_0, reflects the level (or overall intensity) of demand at zero price (i.e., $P = 0$). The span parameter, k, sets the lower bound of consumption (from Q_0) in log units. This model has demonstrated wide generality and precision in fitting demand curves (Hursh & Roma, 2013; 2016). It has been cited by over 500 publications, and both this model and its derivatives (Koffarnus et al., 2015) have become the standard for assessing operant demand in behavioral research. This report provides a novel implementation of the framework introduced by Hursh and Silberberg (2008), reviews the limitations of the EXPL model, and provides a revised approach that resolves these limitations.

Complications of the Exponential Model (EXPL): Hursh and Silberberg (2008)

Despite widespread application in human and nonhuman research (Hursh & Roma, 2013, 2016), there are three overarching limitations to the EXPL model. First, not all forms of consumption can be evaluated in logarithmic units. That is, zero consumption values are undefined on the log scale and must be excluded from analyses. As such, zero consumption values must be either dropped or modified to apply the EXPL model. Second, the log scale has no defined minimum or lower bound. That is, relative decreases on the log scale proceed towards negative infinity and never reach true zero. As a result of this limitation, the authors of the EXPL model included a parameter to restrict the span of the demand curve, namely, k. In practice, this parameter refers to the range of the observed consumption (plus a margin). Often, it is simply treated as a fitted parameter in the range $0 > k \leq 5$ and the best fitting shared value within an experiment is used in subsequent analyses. Ultimately, there are no absolute standards for the value of k and the value typically varies across experiments, complicating comparisons between studies (Kaplan, Foster, et al., 2018). Third, and related to the earlier point, the rate constant α is inherently bound to the span parameter in the EXPL model. That is, the rate of change in consumption is jointly reflected by both α and k (Hursh, 2014). Researchers have typically held k constant within experiments to facilitate comparisons of α, but this approach is not without limitations. For instance, the span of individual demand curves may vary widely across groups and conditions and a single k value may not characterize the data equally well across groups or individual cases. Additionally, comparisons of α across experiments conducted in different laboratories are further complicated because span differences frustrate clear comparisons.

Revisiting the first limitation, the log transformation of consumption introduces challenges when zero consumption values are encountered. As an alternative to excluding data outright, researchers have used several approaches to avoid problematic zero values. Among these, some have recommended replacing zero values with a small, arbitrary constant (Kaplan, Foster, et al., 2018; Koffarnus et al., 2015) and others have restricted analyses to overall aggregates across groups (i.e., consumption was averaged across groups and prices). Further, others have omitted the log transformation of consumption altogether (Koffarnus et al., 2015). Regardless of the approach, few firm guidelines exist for addressing zero consumption values and each of the approaches noted here presents with limitations. For instance, simply dropping the zero values invites the potential for bias in statistical analysis. Koffarnus et al. (2015) noted that the exclusion of observed data not occurring at random has the potential to bias study results because a specific subset of individual responding is intentionally omitted from the analysis. Additionally, researchers have highlighted issues with replacing zero values with small, arbitrary constants when using the log scale (Koffarnus et al., 2015; Liao et al., 2013; Yu et al., 2014). That is, whereas the *absolute* difference between such constants is small (e.g., 0.01 is a 0.09 decrease from 0.1) the *proportional* difference between them is quite dramatic (e.g., 0.01 is a 90% decrease from 0.1). Given that the log scale reflects proportional differences, this can have a significant effect on the span of fitted demand curves and lead to markedly different results depending on the constants used.

Apart from zero consumption values, the log scale limits demand curves analysis in other ways as well. That is, simply finding a way to include zero values in the regression would not address all the limitations noted above. For instance, such a workaround would not remedy the need for an explicit span parameter nor the variability associated with its determination (Kaplan, Foster et al., 2018). Similarly, the rate constant α would also remain coupled to the span and thus incomparable unless span was held constant across cases. That is, the unbounded nature of the log scale thwarts the use of α as a general metric for essential value, the goal of the EXPL model. Hursh (2014) suggested a derivative metric for essential value, EV, that simultaneously considers the combined values of α and k, but that metric has not garnered general acceptance.

Exponentiated Model (EXPD): Koffarnus et al. (2015)

As an alternative to representing changes in consumption on the log scale, Koffarnus et al. (2015) presented a modified form of the EXPL model that evaluated consumption on the linear scale. Briefly, the EXPL model was modified through a process of exponentiating the terms such that changes in consumption were evaluated in the linear, or natural, scale. The form of this model is shown below:

$$Q = Q_0 * 10^{k\left(e^{-\propto Q_0 P} - 1\right)} \qquad (2)$$

This modified model, termed the Exponentiated model of operant demand (EXPD), evaluates changes in consumption using the linear scale (i.e., Q; absolute differences) and supports the inclusion of zero consumption values during regression. That is, consumption is fitted on the linear scale while the span of the demand curve remains reflected on the log scale. Arranged in this way, issues associated with zero consumption values on the log scale are avoided and zero consumption data need not be modified nor excluded during regression.[1]

Although the EXPD model shares the same mathematical basis as the EXPL model (i.e., parameters are identical when fitted to hypothetical exponential demand with zero-error), it warrants noting that the changes in consumption are interpreted differently on linear and log scales. As noted earlier, differences in the log scale are reflective of relative change while differences in the linear scale are reflective of *absolute* change. As such, models derived from the same framework may yield different estimates when changes are evaluated on different scales. That is, differences in the error term are likely to result in varying levels of uncertainty and error variance found to be skewed on the linear scale may present more normally distributed on the log scale.

As noted earlier, issues associated with the EXPL model (and its derivatives) extend beyond accommodating zero consumption values. That is, the restatement applied to derive the EXPD model retains the log scale in the span parameter and issues with the non-zero lower bound remain. Consider the following hypothetical example of the EXPD model fitted with a Q_0 of 282, a k of 2, and an α of 0.0001. In this hypothetical example, numerous zero consumption values exist at high prices (e.g., 1,000, 10,000). With these parameters, levels of demand predicted by the EXPD model at prices of one, 10, 100, 1,000, and 10,000 would be 278, 247, 91, 3.7, and 2.8, respectively. Illustrated here, we see that exponentiation accommodates zero consumption values in the fitting process but the range of predicted demand remains restricted by the

[1] We wish to clarify that zero values are included in the nonlinear regression for the EXPD model but depending on approach these values may be omitted when determining the span of the curve in log units.

span parameter in log units, k.[2] That is, the lower bound for the EXPD model cannot be asymptotic at zero because zero is undefined on the log scale. Mathematically, the inability to reach a lower bound at zero is to be expected because the EXPD model shares the same functional form of the EXPL model. That is, the underlying rate of change (α) is represented jointly with the span parameter and this span parameter for both models is defined in log units (i.e., log range of upper and lower nonzero consumption). As a result, the asymptotic minimum of both the EXPL and EXPD models at high prices can never be zero consumption because the underlying log scale decreases to negative infinity.

Relative vs. Absolute Model Error

With respect to performing nonlinear regression, we note that residual error in log scale is more accurately described as the (root) mean squared *percentage* of error rather than the more typical (root) mean squared error. Differences between the log and linear scales here are made clearer by restating one of the rules of logarithms: The logarithm of a quotient is the difference between two separate logarithms and the calculation of residual error on the log scale represents a difference of logarithms. The EXPL model evaluates consumption in the log scale; therefore, the residual error is a function of relative differences and relative change is at the heart of elasticity. Additionally, beyond supporting a unitless estimate of changes in Q, relative error supports demand curve analyses in other ways as well. For instance, residual error determined in this way remains comparable when levels of consumption vary across both high and low levels. That is, the root mean squared residual error is proportionally consistent at extremes (i.e., normalized) while the absolute root mean squared residual error may or may not be.

Residual error representation becomes increasingly relevant in situations where demand for a reinforcer is simultaneously evaluated across different prices as well as different magnitudes (e.g., dose, concentration). One method for evaluating demand is to simultaneously fit individual demand curves across prices and various levels of reinforcer magnitude. The results of which provide dosage specific Q_0 parameters and represent overall sensitivity to price with a single, shared α parameter. In the EXPL model, error variance at each dosage level jointly contributes to the overall error variance in a comparable fashion when the individual Q_0 values are observed across multiple orders (e.g., 1, 10, 100). Shown below in Equation 3 is a representation of the percentage error in the log scale.

$$\begin{gathered}\hat{Y}_n = log_{10} Q_0 + k\left(e^{-\alpha Q_0 X} - 1\right) + \varepsilon_n \\ \varepsilon_n = log_{10} Y_n - \hat{Y}_n \\ \varepsilon_n = log_{10} \frac{10^{\hat{Y}_n}}{10^{Y_n}}\end{gathered} \tag{3}$$

In contrast to relative error, nonlinear regression using the EXPD model is driven using absolute error (i.e., mean squared error). This distinction is worth highlighting because the mean squared error has the potential to disproportionally contribute to overall error variance. That is, demand evaluated simultaneously across prices and magnitudes yields residual error that naturally grows as a function of demand intensity (i.e., Q_0). As a result, the degree of error variance across these related demand curves differentially contribute to the overall error variance. This has an unintended effect of occasionally frustrating the characterization of demand across magnitudes with a singular rate parameter. As such, it is possible and likely under certain circumstances that the two models could ultimately lead researchers to different conclusions because of the differences in how error variance is reflected.

Alternative Log-like Transformations

Difficulties associated with zero values and the log scale extend beyond applications in operant demand and researchers across various fields have explored alternative scales and transformations to handle these situations. Although many such transformations exist, for instance the Box-Cox (BC) family of transformations (Box & Cox, 1964), this report highlights the Inverse Hyperbolic Sine (IHS) transformation and how it has been used successfully to accommodate zero consumption

[2]Mathematically, the nonzero lower asymptote for the EXPL/EXPD model is represented by the difference in orders of magnitude, that is, $10^{\log_{10} Q_0 - k}$.

values in economic analyses (Burbidge et al., 1988; Johnson, 1949). Relative to other transformations (e.g., BC), the IHS transformation is particularly desirable because it natively supports log-like transformations for positive, negative, and zero values.

The use of the IHS transformation was proposed in Johnson (1949), but its utility in areas of economics was only more recently reviewed in Burbidge et al. (1988) and Bellemare and Wichman (2019). As noted in these works, the IHS transformation is favorable to economic research because of its "log-like" form and its predictable behavior at and below zero. Regarding its mathematical basis, the IHS (i.e., $sinh^{-1}$) or area sine hyperbolic (i.e., *arsinh* or *asinh*; *ar-* or *a-* to emphasize area rather than arc), is an inverse of hyperbolic functions. That is, whereas *sin* is defined in terms of the unit circle, *sinh* is defined in terms of the unit hyperbola. Given these differences, the inverse of these functions naturally yields distinct interpretations. Specifically, the inverse of the *sin* (i.e., *asin*) function yields the length of the arc of the unit circle and the inverse of the *sinh* (i.e., *asinh*) a measure of area.[3]

As a means of transformation, the IHS scale is particularly versatile in that it is simple to calculate, is flexible and customizable, and is not undefined at zero: $sinh^{-1}(0) = 0$. That is, the IHS transformation is easily applied and can be performed in several ways (see Eq. 4).

$$Y = asinh(X) = sinh^{-1}(X) = ln\left(X + \sqrt{X^2+1}\right) \tag{4}$$

Although applicable in its most brief form (*asinh*), the IHS transformation may be adjusted to emulate the properties of another scale or transformation. For example, this method can be adjusted such that it becomes more approximate to oft-used log scales, for example, natural log (Mount, 2012). This form taken by this generalized IHS transformation is provided in Equation 5.

$$Y = ln\left(\theta X + \sqrt{\theta^2 X^2 + 1}\right) \tag{5}$$

Using parameter θ in Equation 5, the IHS transformation can be adjusted to closely emulate the log scale over most of its range. Once a suitable θ has been determined, the final transformation behaves highly approximate to the desired scale. In our efforts to closely emulate the $\log_{10}$ transformation, we have found a θ of 0.5 to be suitable to approximate the scale. Further discussion of our optimization of θ and determination of this specific value is provided in the Appendix. The final form of this $\log_{10}$-like IHS transformation is provided in Equation 6.

$$\begin{gathered} Y = \frac{ln\left(.5x + \sqrt{.25x^2+1}\right)}{ln(10)} \\ Y = log_{10}\left(.5x + \sqrt{.25x^2+1}\right) \end{gathered} \tag{6}$$

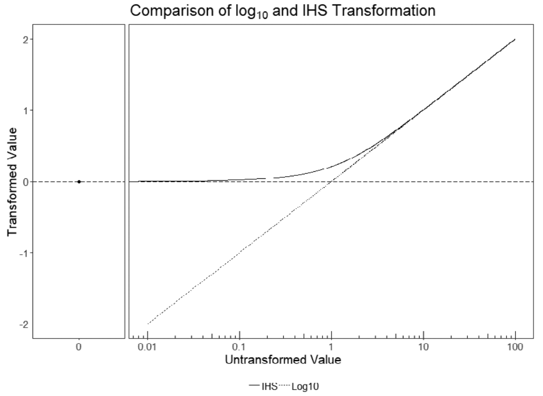
The transformation in Equation 6 is well-suited for use in the operant demand framework because it emulates the $\log_{10}$ scale across most values and accommodates zero consumption values. Illustrated in Figure 1, the transformed and natural values across the $\log_{10}$ and IHS scales are illustrated across values ranging from 0 to 100. Shown in this figure, the $\log_{10}$-like IHS scale is virtually identical to the $\log_{10}$ scale at values of 10 or above and gradually approaches zero for values less than 10.[4] Although the progression from 10 to 0 naturally differs from the $\log_{10}$ scale being emulated, the progression from 10 to 0 continues to correspond with decreases observed in the log scale. That is, for a data value of 5, there is only a 2% difference between $\log_{10}$ and IHS transforms, and at 3 less than a 10% difference. Hence, the major differences between scales are most represented with values of 1 and below. Lastly, transformed values may be returned to their original values using Equation 7.

$$X = \frac{1}{10^Y}\left(10^{2Y} - 1\right) \tag{7}$$

[3]We note that the area result for *asinh x* is derived from the (doubled) area that falls between the x-axis, a ray passing from the origin (0, 0) to a relevant point of reference (*cosh x*, *sinh x*), and the unit hyperbola.

[4]We note here that the index for $\log_{10}$-like IHS scale is 10 and that transformed values above the index are essentially identical between the $\log_{10}$ and $\log_{10}$-like IHS scales.

Figure 1
Comparison of the Log_{10} and IHS Transformations



Note. This figure illustrates how each transformation corresponds across a range of consumption values, including zero. The $\log_{10}$ transformation changes dramatically as it approaches zero while the modified IHS transformation is approximate to the $\log_{10}$ at values ~10 and smoothly approaches zero at lesser values.

A Zero-Bounded Model of Operant Demand

The log scale has been both an asset and a challenge in demand curve analysis. Zero consumption values are undefined in this scale and the log-based span (k) used in both the EXPL and EXPD models does not support a zero asymptote (i.e., a true lower bound at zero). Approaches to mitigating these limitations have been the source of significant debate and complete solutions to these issues with the log scale have not yet emerged. The IHS transformation presented here provides a means to resolve these challenges while retaining the functional form of the EXPL and EXPD models, namely, the framework presented in Hursh and Silberberg (2008). That is, the original framework and interpretation are preserved, and the typical $\log_{10}$ scale is replaced with a "$\log_{10}$-like" alternative. The direct restatement of the EXPL model using the IHS scale is provided in Equation 8.

$$IHS(y) = IHS\left(Q_0\right) + IHS\left(Q_0\right) * e^{-\propto Q_0 x} - 1 \quad (8)$$

where $IHS\left(Q_0\right) = log_{10}\left(0.5\, Q_0 + \sqrt{0.25\, Q_0^2 + 1}\right)$

This restatement of the EXPL model, hereafter referred to here as the Zero-Bounded Exponential model (ZBE), evaluates changes in consumption in the $\log_{10}$-like IHS scale rather than the $\log_{10}$ (or linear) scale. This is presented as a novel implementation of the framework because differences in scale support new behavior. First, the ZBE model can reach a true lower bound of zero (i.e., it can accommodate zero and nonzero asymptotes) and this will never be possible when the span of consumption is reflected in log units. Specifically, consumption represented in IHS units has a maximum at IHS(Q_0) and a minimum of zero; hence, the span of the demand curve in IHS units is simply IHS(Q_0). As a result, Q_0 can serve as its span and this obviates the need for a separate span parameter (or any additional margin) in most cases. Second, the span of the demand curve is far less sensitive to consumption occurring at very low rates (i.e., fractional). For instance, several applications of the ZBE model are illustrated in Figure 2 with a trailing zero consumption value retained, replaced with 0.1, and replaced with 0.01. Shown here, fittings across these conditions produced well-fitting curves with estimates that were virtually identical regardless of the zero or near-zero values included. That is, the small constants used to replace zero (e.g., 0.1, 0.01) did not produce the undesirable variability rightly highlighted in Koffarnus et al. (2015).

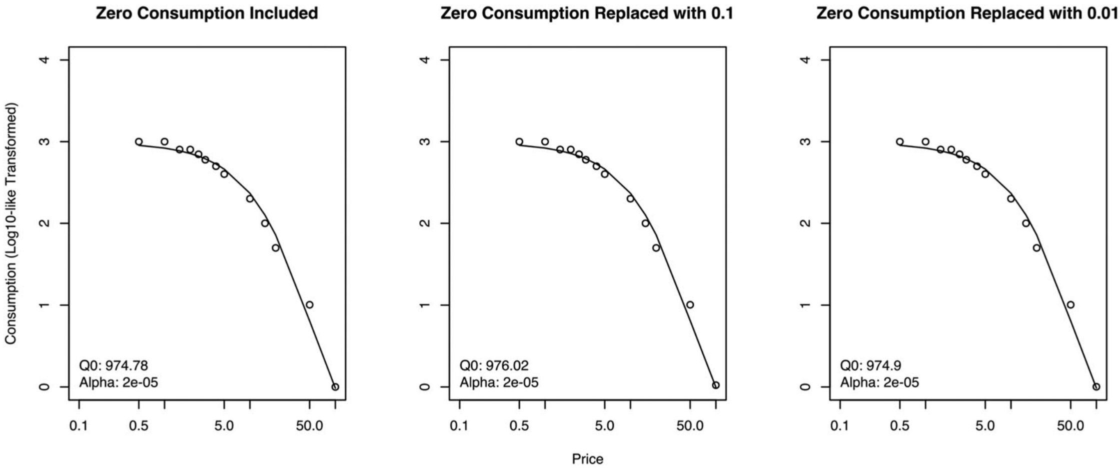
As noted earlier, the ZBE model can be simplified by removing the explicit span parameter, k. As noted by Hursh (2014), α varies with span (k) in the EXPL model and this frustrates comparisons in cases where span parameters vary. In removing parameter k and normalizing α to the units of IHS*(Q_0)*, the ZBE model effectively removes this coupling. Factoring out the duplicated span term in Equation 8 and dividing α by *IHS(Q_0)*, this yields a normalized version of the ZBE model (ZBEn). This is shown in Equation 9 and further information on this derivation is provided in the Appendix.

$$IHS(y) = IHS\left(Q_0\right) * e^{-\frac{\propto}{IHS\left(Q_0\right)} Q_0 x} \quad (9)$$

The goals of this report were to introduce the IHS scale as a replacement for the log scale in the Hursh and Silberberg (2008) framework, to review the benefits and applicability of the two forms of the ZBE model (i.e., ZBE, ZBEn), and to evaluate the performance of the ZBE models across both simulated and peer-reviewed data. Specifically, derived measures of

Figure 2

IHS Transformation across Zero-valued Data Situations



Note. These plots illustrate how the IHS transformation makes the ZBE model robust at zero and near-zero consumption. In each fitting, model results remain consistent across conditions where zero consumption data are retained or modified.

intensity (Q_0) and P_{MAX} from various models were compared to their observed, empirical equivalents, $Q_{0\text{-}E}$ and $P_{MAX\text{-}E}$. Additionally, each model was evaluated with respect to its ability to characterize demand for a reinforcer across multiple levels of magnitude (i.e., doses). Consistent with earlier work in this domain (Gilroy & Kaplan, 2019), all data-generating processes and analytical syntax have been publicly archived in a GitHub repository hosted by the corresponding author under an open-source license for public inspection and future replication.[5]

Method

Simulated Data: Hypothetical Purchase Task

A total of 1,000 simulated consumption series with and without zero values (48.39% zero values) were generated using seed parameters and error variance derived from a peer-reviewed article related to operant demand (Koffarnus et al., 2015). Whereas the focus of Koffarnus et al. (2015) was to recover individual model parameters (e.g., Q_0, α) in the EXPL and EXPD models, the goal of the current simulation was to evaluate the correspondence between model estimates to observed complements. That is, the measures from each model were evaluated against empirical indicators of demand (i.e., Q_0 to $Q_{0\text{-}E}$ and P_{MAX} to $P_{MAX\text{-}E}$). Empirical measures were used as the primary indicators of demand to facilitate consistent comparisons across models. Specifically, $Q_{0\text{-}E}$ was simply the level of observed consumption at the lowest price and $P_{MAX\text{-}E}$ was derived by determining the price associated with maximum observed output, $Q * P$. Parameter α was not directly evaluated, given that differences were to be expected between models due to scale differences rather than the performance of each model.

All simulation, model fitting, and calculations were performed using the R Statistical Program (R Core Team, 2017). Simulation data series were prepared using the group-level estimates and residual error variance reported in Koffarnus et al. (2015).[6] The prices included in the simulation consisted of the following:

[5]The full source code and materials necessary to replicate all analyses is archived at https://github.com/miyamot0/ZeroBoundedDemand

[6]Simulation scripts designed to replicate simulation procedures used in Koffarnus et al. (2015) were adapted from the *beezdemand* package (Kaplan, Gilroy et al., 2018). Seed values for the simulations were Ln α (SE) = -2.5547 (0.7025), Ln Q_0 (SE) = 1.239 (0.3202), k = 3.096, and Y_{SD} = 1.438.

\$0.10, \$1.00, \$3.00, \$10.00, \$30.00, \$100.00, \$300.00, and \$1000.00. These parameters and prices were used to generate series representative of the original data, and simulated data that passed criteria for systematic purchase task data were included in subsequent analyses (Stein et al., 2015).

Simulated Data Analytical Plan

Simulations were designed to evaluate demand curve metrics, across models and varying compositions of data. To minimize differences due to varying numbers of parameters (i.e., 2 vs. 3), formulations of both the EXPL and EXPD models were prepared using parameter Q_0 in place of parameter k. In this way, the span as represented in the direct restatement ZBE model was reflected in the EXPL and EXPD models as well. That is, the theoretical span across all models was made more comparable by referencing the individual intensity of demand (Q_0) for each respective model.[7] Both the original and modified forms of each model are indicated in Table 1. Removing the potential variability due to varying numbers of parameters, fitted Q_0 parameters and derived P_{MAX} calculations from each of the models were then compared to their empirical complements using Spearman correlations.

Parameter Estimation and Calculations of P_{MAX}

Measures of P_{MAX} were calculated for each of the models of operant demand. The EXPL and EXPD models were fitted using methods derived from the *beezdemand* package in the R statistical program (Kaplan, Gilroy et al., 2018). Model fitting for the ZBE model was also performed using methods derived from *beezdemand*, customized to support the newer model. Although calculations of P_{MAX} can be performed using exact solution methods for both the EXPL and EXPD models (Gilroy et al., 2019), P_{MAX} for the EXPL, EXPD, and ZBE models was determined iteratively by directly evaluating work output across prices in the natural scale. That is, P_{MAX} was determined from the fitted model and was consistent with an Observed P_{MAX} approach but instead evaluated fitted model predictions (Gilroy et al., 2020; Greenwald & Hursh, 2006). Although methods can be used to determine unit elasticity with ZBE via a derivative approach, it warrants noting that rates of change in the IHS scale present challenges with consumption values that exist well below the index value, for example, 1 (Bellemare & Wichman, 2019). That is, the aspect of the ZBE that supports the zero asymptote also influences traditional calculations of unit elasticity when prices and/or consumption are nearer to zero. Put simply, the transformed units are not constant below the index and this inconsistency renders a slope of −1 questionable as a correlate to the price of max responding, P_{MAX}. As such, researchers could optimize for the price value closest to the elasticity of −1 in IHS–IHS coordinates but this price is unlikely to consistently correspond with peak levels of responding (i.e., O_{MAX}). In situations where the levels of consumption or price are below the index value, the conservative approach to determining P_{MAX} would be to optimize for the price that yields the highest levels of work in the natural scale (i.e., Observed P_{MAX}) or evaluate changes in consumption and price as percentages (both in natural units).[8] Both approaches are applicable for all models but the Observed P_{MAX} was selected given that it was the more conservative approach.

Published Data: Dosage-level Effects

Peer-reviewed data from Ko et al. (2002) and Winger et al., (2002) were obtained from one of the authors (S. R. Hursh) and reanalyzed. These studies, specifically, were selected due to their inclusion in the earlier validation of the Hursh and Silberberg (2008) framework and were discussed in Hursh and Roma (2013, 2016). These studies included three adult rhesus monkeys as the primary subjects. The Ko et al. (2002) study included analyses comparing the relative reinforcing effects of three μ-opioid agonists (fentanyl, alfentanil, and remifentanil). These data are useful for

[7]A small constant is typically added to the span in the EXPL and EXPD models and 0.5 was added to the estimated span in each of the respective fittings.

[8]Unit elasticity is calculated as relative change in consumption (Q) divided by relative change in price (P), both in natural units. Tools for calculation of P_{MAX} and O_{MAX} are available at https://www.smallnstats.com/index.php?page=ZBE and https://ibrinc.org/behavioral-economics-tools/.

Table 1

Structure of Individual Model Candidates

Model	Three-Parameter Form	Two-Parameter Form
EXPL	$log_{10}Q_0 + k(e^{-\propto Q_0 x} - 1)$	$log_{10}Q_0 + log_{10}Q_0(e^{-\propto Q_0 x} - 1)$
EXPD	$Q_0 * 10^{k\left(e^{-\propto Q_0 x} - 1\right)}$	$Q_0 * 10^{log_{10}Q_0\left(e^{-\propto Q_0 x} - 1\right)}$
ZBE	$IHS\left(Q_0\right) + k(e^{-\propto Q_0 x} - 1)$	$IHS\left(Q_0\right) + IHS\left(Q_0\right) * (e^{-\propto Q_0 x} - 1)$
ZBE (Normalized $\propto$)	$IHS\left(Q_0\right) + k\left(e^{-\frac{\propto}{k} Q_0 x} - 1\right)$	$IHS\left(Q_0\right) * \left(e^{-\frac{\propto}{IHS\left(Q_0\right)} Q_0 x}\right)$

Note. Table 1 depicts 2- and 3-parameter model structures across the three candidate models. Parameter k in respective models represents the range of demand in units of the respective scale (i.e., log, IHS). The normalized α form is presented as a method to support model evaluation across multiple demand levels when demand intensity varies with doses of a drug or magnitudes of reinforcement.

evaluating demand models because three dosages of each drug were evaluated and the potencies of the drugs spanned a two-log range. The design of the Hursh and Silberberg (2008) framework is such that the Q_0 parameter controls for differences in dose and potency (reinforcer magnitude) and the α parameter reflects sensitivity to price in conjunction with the span parameter, k. At the same time, the three opiates all have rapid times to onset of action and should be very similar in value as reinforcers and have equal or nearly equal α terms as reported by Hursh and Roma (2013) using the EXPL model.

Data from Winger et al. (2002) were also obtained to evaluate relative reinforcing effects across agonists. Specifically, this study evaluated the reinforcing effects of three NMDA agonists (ketamine, phencyclidine, and dizocilpine). These three drugs were selected because they have distinctly different delays to onset of peak pharmacological action and the original study indicated that this factor rank ordered the drugs in terms of P_{MAX}. In a subsequent analysis using the EXPL model, Hursh and Roma (2016) reported that the αs also rank ordered the drugs based on time to peak action. As in Ko et al. (2002), three doses of each drug were used and ideally the α term would be constant across doses but vary between drugs and different times to peak action.

Original data from both studies were used to evaluate how well each model isolated variations in reinforcer magnitude (e.g., dose) using respective Q_0 parameters while simultaneously representing the sensitivity to price with a single, shared α parameter. In addition to replicating earlier group-level analyses (which averaged zero and nonzero consumption across prices), data at the individual-case level were revisited with the IHS scale. This exploratory modeling of individual consumption was pursued as a complement to group-level analyses, in this regard, to illustrate how the ZBEn model performed with the complete case-level data previously unusable in the original framework.

Analytical Plan: Dosage-Level Effects

Data from Ko et al. (2002) and Winger et al. (2002) were reanalyzed using a GraphPad Prism 8 template customized to apply candidate models. In contrast with the earlier simulation, the empirical data were modeled with the ZBEn model (Eq. 9) to uncouple the α parameter from differences in span, see Table 1. Individual-level data were evaluated using the EXPL, EXPD, and ZBEn (Eq. 9) models. These individual data contained some zero consumption values; specifically, the Ko et al. data set consisted of 2.2% zero consumption values (3/135) and the Winger et al. data set consisted of 9.4% zero consumption values (15/160). As indicated in Winger et al., due to an oversight during the experiment, data points for two monkeys were missing (dizocilpine 0.003 mg/kg at FR1). In keeping with existing conventions, parameter k was included in the EXPL and EXPD models, and this value was fitted along with the other fitted parameters. This behavior was retained to compare the performance of ZBEn to existing models consistent with their current and historical applications. For models including parameter k, this parameter was fitted globally. Although not without its issues, the R^2 was reported as a general goodness-of-fit metric given that each model was fitted on a different scale. Specifically, each R^2 value reflected variability for each of the monkeys within each drug.

To evaluate the degree to which demand for each drug across doses was accommodated by a single, shared α parameter, models were compared using the Akaike's Information Criteria (Akaike, 1974) corrected for small sample sizes (AICc). The comparison logic of AICc is like that of the Extra-Sum-of-Squares F-test and evaluates changes in the residual sum-of-squares against differences in the degrees of freedom. That is, the model with varying α parameters has fewer degrees of freedom than the model with a shared α parameter. AICc was preferred because it yields a probability measure, rather than a p-value, to describe the likelihood the data were generated from each respective model.

Results

Simulation Study: Koffarnus et al. (2015)

The overall distributions of P_{MAX} and intensity (i.e., Q_0) derived from each model are illustrated in Figure 3. Overall, P_{MAX} and Q_0 derived from each of the three 2-parameter models were similar to their empirical alternatives. Regarding P_{MAX}, Spearman correlations between empirical and derived estimates were overall strongest when zero consumption values were omitted in the EXPL model ($r_s = 0.695$, $p < .0001$). Within the models that included zero values, the relationships between empirical and derived estimates of P_{MAX} were stronger for the direct restatement of ZBE model ($r_s = 0.389$, $p < .0001$) than the EXPD model ($r_s = 0.268$, $p < .0001$). Similarly, correlations between empirical and derived estimates of demand intensity were strong for the EXPL ($r_s = 0.942$, $p = 0$) and ZBE models ($r_s = 0.848$, $p = 0$) models but strongest for the EXPD model ($r_s = 0.962$, $p = 0$).

Reanalysis of Ko et al. (2002)

Data from Ko et al. (2002) were used to evaluate the demand for three rapidly acting opiates with a similar time to onset of action. These data were reanalyzed with each of the three model candidates and the results from each are illustrated in Figure 4. Results from AICc for data fitted using the ZBEn model suggested a single α for alfentanil (87.33%) and fentanyl (64.73%) but did not accommodate a single α for remifentanil (25.28%). Fits to the available data revealed that the EXPL did not accommodate the different doses for any of the three drugs. The probability the data were generated from a single α was 48.47%, 0.17%, and 19.12% for alfentanil, remifentanil, and fentanyl, respectively. Finally, fits using the EXPD suggested this model accommodated a single α for the different doses of all three drugs. Specifically, results from AICc indicating the data were generated from a single α was 94.50%, 66.66%, and 93.22% for alfentanil, remifentanil, and fentanyl, respectively.

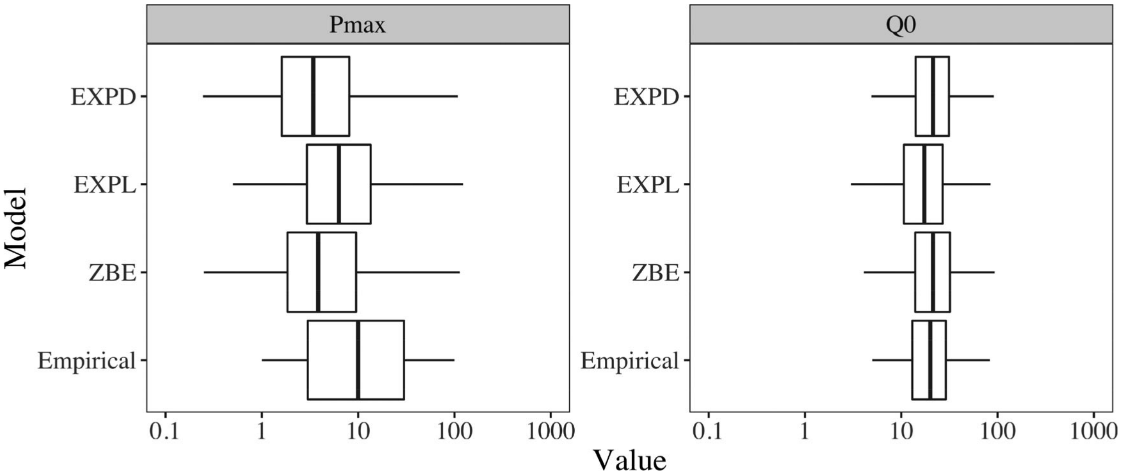
Reanalysis of Winger et al. (2002)

Data from Winger et al. (2002) were gathered to evaluate the demand for three NMDA agonists that differed in their delay to peak onset. These data were reanalyzed with each of the three model candidates and the results from each of the models are illustrated in Figure 5. Results of AICc for the ZBEn revealed the model accommodated a single α for ketamine (85.05%), phencyclidine (82.68%), and dizocilpine (73.58%). Fits to the available data revealed that the EXPL accommodated a single α for ketamine (94.04%) and phencyclidine (90.29%), but not dizocilpine (43.86%). Finally, results from AICc for the EXPD revealed the model accommodated a single α for ketamine (93.38%) and phencyclidine (91.90%), but not dizocilpine (0.13%). All three models preserved sensitivity of α to differences across drugs in terms of time to peak effect (Figure 5), wherein ketamine (time to peak effect: 1 min) resulted in the smallest αs, phencyclidine (time to peak effect: 10 min) resulted in intermediate αs, and dizocilpine (time to peak effect: 32 min) resulted in the largest αs.

Case-level Analyses for Ko et al. (2002) and Winger et al. (2002)

The Ko et al. (2002) data included three drugs and three doses across three primates. This provided nine dose-families of demand curves for the three drugs and three subjects. Using the ZBEn model and GraphPad Prism 8, the αs for all nine sets of curves were similar across doses (AICc range from 99.9% to 71.6%). In addition, the αs were similar for the three opiates; α for alfentanil and remifentanil were similar within subjects and lower than α for fentanyl, consistent

Figure 3
Simulated Comparisons from Koffarnus et al. (2015)



Note. This figure illustrates the overall distribution of values associated with P_{MAX} and intensity (Q_0). The IHS transformed model consistently supported estimates that closely related to the empirical data.

with the previous analysis in Hursh and Roma (2013).

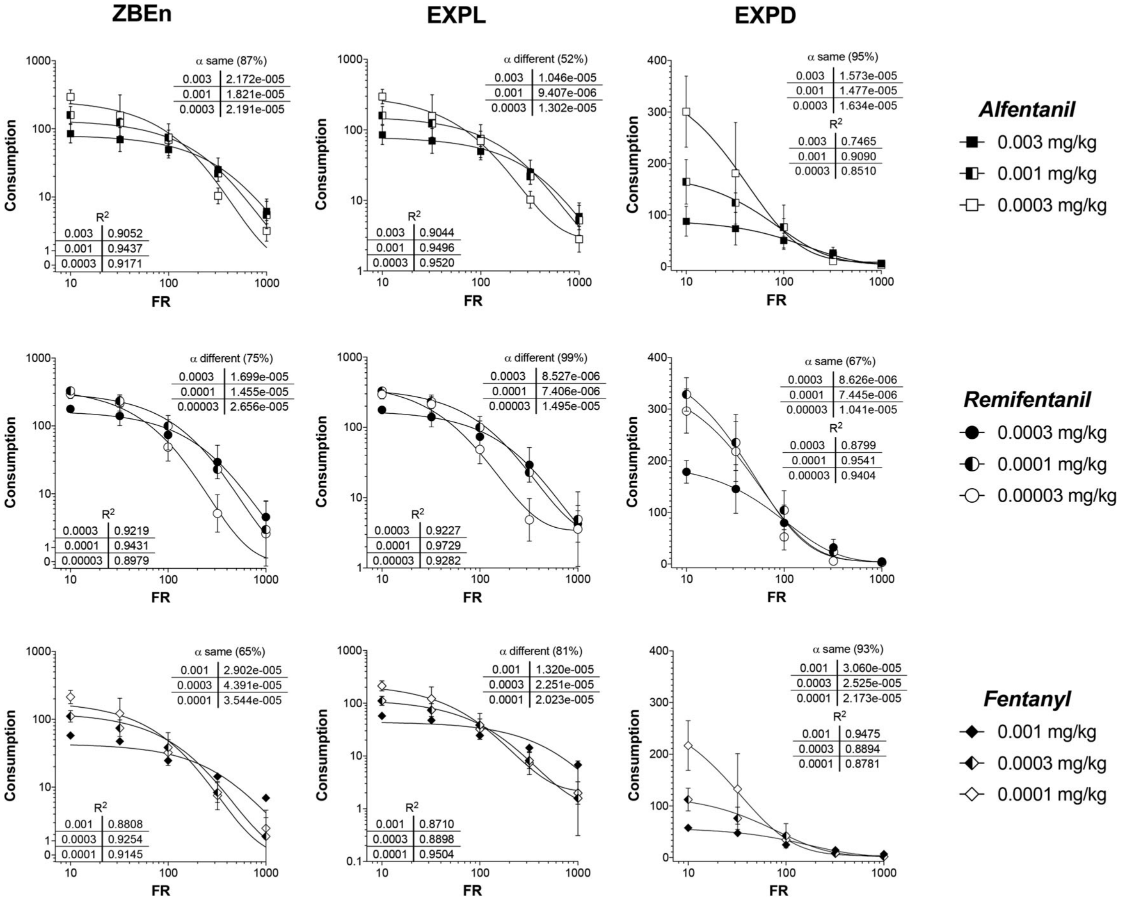
The Winger et al. (2002) study evaluated NMDA agonists (ketamine, phencyclidine, and dizocilpine). As in Ko et al. (2002), this resulted in nine dose-families of demand curves for three drugs and three subjects. Using the ZBEn model, the αs for all nine sets of curves were similar across doses (AICc range from 97.7% to 59.6%). In addition, the αs were ordered in increasing α (sensitivity to price) from ketamine, to phencyclidine, to dizocilpine, consistent with the previous analysis in Hursh and Roma (2016). Although all 18 demand curve comparisons from these two studies showed consistent values of α across dose, it warrants noting that these exploratory case-level evaluations lack sufficient power to detect small differences in the α term and do not account for intrasubject correlations.

Discussion

The behavioral economic concept of demand is increasingly applied across disciplines and to various types of consumer choices. Along with this more diverse application, the types and compositions of data used in this framework have grown as well (e.g., hypothetical purchase tasks). However, the existing models for characterizing operant demand each present with respective challenges and these challenges are barriers to future applications. Among these challenges, debates continue regarding how to handle zero consumption values while retaining the underlying log scale in the framework put forth in Hursh and Silberberg (2008). For instance, advocates of the EXPD model have encouraged the retention of the log scale to represent the span of the demand curve (i.e., *k*) but the use of the linear scale to evaluate changes in consumption. In this way, zero consumption values may be included in the regression. Although this approach has proved beneficial (González-Roz et al., 2019; Strickland et al., 2016; Strickland et al., 2019; Zvorsky et al., 2019), the exponentiation of terms in the EXPL model does not provide a full solution to the challenges introduced by the log scale. For instance, models of demand using the log scale are restricted to nonzero asymptotes and observed consumption at very high prices is likely to reach zero. The goal of this report was to introduce a novel implementation of the Hursh and Silberberg (2008) framework that used the IHS scale in lieu of the log scale. The use of the IHS in operant demand mirrors trends in statistics and economics, as this scale has been used in economic analyses to accommodate zero values (Bahar & Rapoport, 2018; Clemens & Tiongson, 2017).

The ZBE model applied here performed well across a variety of compositions of data. Specifically, the ZBE model performed well

Figure 4
Ko et al. (2002) *Study Data using ZBEn, EXPL, and EXPD Models*

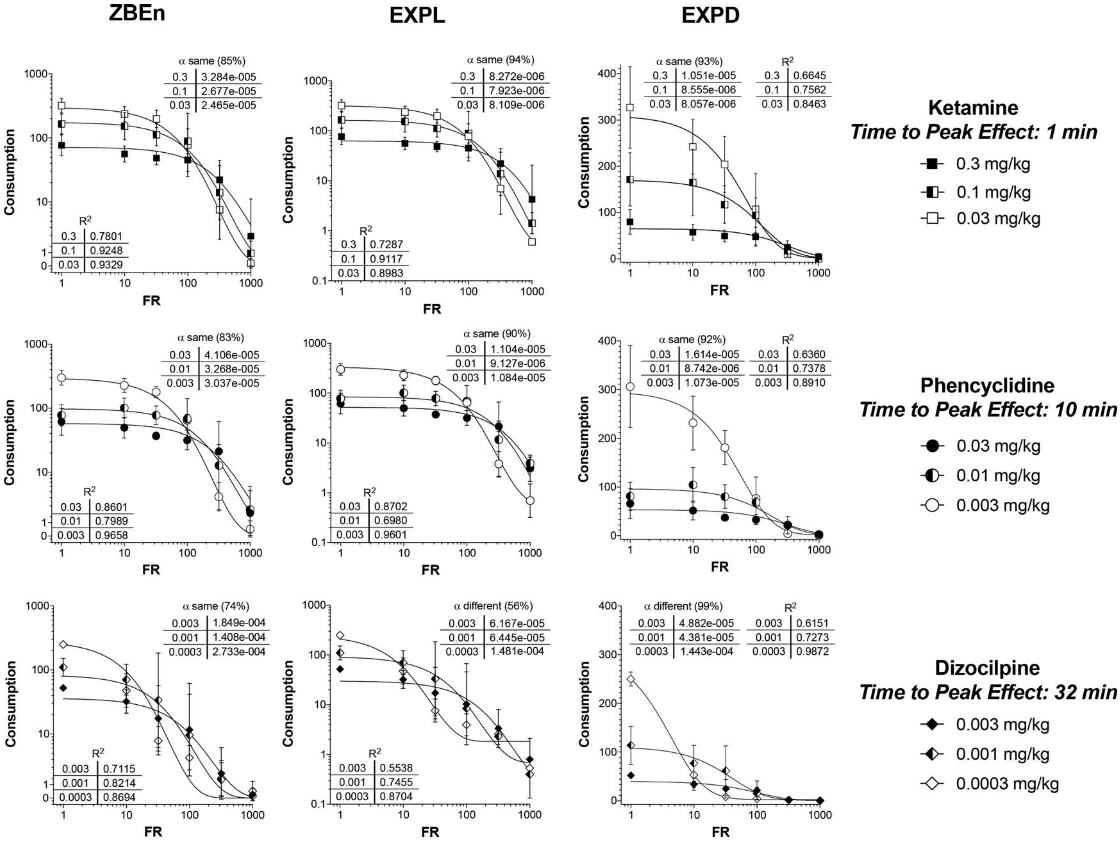


Note. This figure illustrates how each model accommodates demand curve analysis across dosage levels for each of the three drugs. Results from AICc indicate whether the test preferred the model with a same alpha or the model with different alphas. The percentages indicate the probability associated with that conclusion. Results of the AICc comparisons suggested the EXPL model was unable to describe doses using a single α for any of the drugs; the EXPD model was able to describe doses using a single α for all three drugs; and the ZBEn model was able to describe doses using a single α for alfentanil and fentanyl, but not remifentanil.

across data consistent with hypothetical purchase task data as well as those observed in basic lab experiments. In simulations designed to resemble the variable responding frequently observed on hypothetical purchase tasks, the ZBE model provided estimates that corresponded well with empirical estimates of P_{MAX}. Similarly, the ZBE model provided estimates of demand intensity that corresponded well with the observed data. However, it warrants noting that the ZBE model showed slightly lower correspondence than the EXPD model in this regard. Similarly, reanalyses of basic research revealed that the ZBEn model with normalized α and the EXPD model represented sensitivity to changes in prices with a shared α across most combinations of dosages. At the same time, normalized α values remained sensitive to differences in the value of drugs as reinforcers, such that the value of α increased (more sensitive to price effects) with longer times to peak effect, see Figure 5.

Successful demonstrations using the ZBE and ZBEn models naturally evoke questions regarding how researchers should analyze operant demand data moving forward. As with any statistical decision-making, the analytical approach should be carefully selected and

Figure 5
Winger et al. (2002) *Study Data using ZBEn, EXPL, and EXPD Models*



Note. This figure illustrates how each model accommodates demand curve analysis across dosage levels for each of the three drugs and relative magnitude of the α across drugs that differ in their delay to peak effects. Results of the AICc comparisons suggested the EXPL model was able to describe doses using a single α for ketamine and phencyclidine, but not dizocilpine; the EXPD model was also able to describe doses using a single α for ketamine and phencyclidine, but not dizocilpine; and the ZBE-n model was able to describe doses using a single α for all three drugs. All three models revealed relatively orderly differences in α related to time to peak effect with smallest αs for ketamine, intermediate for phencyclidine, and largest for dizocilpine.

evaluated against existing alternatives. Among the strengths of this novel approach, a $\log_{10}$-like scale is exceptional in that it is capable of both accommodating and modeling consumption at zero. The ZBE models are alone in this regard because both the EXPL and EXPD models are ultimately limited by the underlying log scale. Free of this limitation, the ZBE models may be further extended to applications where questions of zero consumption are a major factor. For instance, the ZBE model might be useful in cross-price demand assays to predict the price at which a base commodity is completely substituted, that is, consumption of that good is fully suppressed. Although not explicitly modeled here, the IHS scale and application is suitable for use in cross-price analyses though the same caveats apply (i.e., changes below the index behave differently than those above the index). Scenarios such as the one noted here support the use of the ZBE models, though outside of such specialized questions it is plausible that existing models might be more easily applied and interpreted.

Although we report findings that the three models tested here (EXPL, EXPD, ZBE) all performed well across simulated and experimental data (to varying degrees), we suggest researchers consider the ZBE model for several reasons. First, the ZBE model succeeds in both fitting and modeling zero consumption. That is, the ZBE model extends the operant demand framework by providing means to achieve a lower asymptote at zero. When plotted in IHS–IHS coordinates, this defines an S-shaped demand curve originally suggested by Hursh and Silberberg (2008). This refers to an upper bound at Q_0 and an absolute lower bound at either a zero or nonzero asymptote. The IHS scale furthers this intent of this framework because log units restrict all projections of demand to nonzero asymptotes. Although Hursh (2018) made the case for nonzero asymptotic demand for certain luxury goods, such as fractional ownership of executive jets and leasing of limousines, the problem with modeling demand using the log scale is that we must presume that all commodities have nonzero lower asymptotes. Whereas this may not be true for most goods, considering that there are often cheaper alternatives that can and will drive demand to zero at high prices, there are instances where goods are demanded well above the prices assessed. Among such scenarios, the ZBE has equal utility (i.e., a separate span parameter can be included as needed).

In a more pragmatic sense, the ZBE model supports a more flexible implementation of the Hursh and Silberberg (2008) framework that requires fewer parameters. For instance, in cases where levels of consumption reach zero, the full range of consumption values is ultimately the intercept (i.e. demand intensity – Q_0). This removes the need for a span parameter in these instances. In these situations, the IHS scale supports an interpretation of demand that can exist free from parameter k and the variability it introduces in research (Kaplan, Foster, et al., 2018). Although removing the explicit span parameter often simplifies parameter estimation, comparison, and interpretation, situations may exist when an explicit span parameter is beneficial. That is, Q_0 serves as an excellent substitute for k when the full range of the demand curve ranges from Q_0 to zero but this may not be the case when the span of the demand curve falls far short of zero. In such situations, the ZBE model can be adjusted to include an explicit span parameter as necessary. Furthermore, model complexity can be formally evaluated on a case-by-case basis via traditional model selection tools (e.g., AICc). Second, the normalization of α decouples its relation to k and instead couples it to the case-level Q_0 parameter. This was demonstrated in the reanalyses of drug dosage data, as the normalized α supports comparisons of demand even in instances where the spans differ (i.e., there was no shared span value). However, it warrants noting that this has the added effect of limiting comparisons of Q_0 and α to earlier models (e.g., EXPL, EXPD). That is, parameter estimates from the ZBEn model are unlikely to remain as closely related to other models when a normalization is applied and any comparisons should proceed with caution.

Lastly, the ZBE model and the underlying IHS scale behave sufficiently $\log_{10}$-like that differences associated with the error term are minimized, if not eliminated. That is, the behavior of the error term is consistent with the original implementation of the Hursh and Silberberg (2008) framework, such as evaluating demand across magnitudes (e.g., dose). As such, the $\log_{10}$-like scale presented here provides the most desirable features of the log scale without the issues associated with zero values. We note here that evaluations across reinforcer magnitudes can succeed in the linear scale; however, error estimates in this state are likely to vary considerably across magnitudes and occasionally provide results that differ from the EXPL and ZBE models.

Limitations and Further Research

This report presented and evaluated a method for evaluating operant demand using a $\log_{10}$-like scale, the IHS transformation. Although the log scale is an effective tool in many areas of economics, its use in operant demand research is often frustrated by zero values. For this reason, an exploration of alternative scales is both timely and appropriate. While the IHS transformation is promising, it warrants reiterating that the transformation presented here is ultimately one of several that could extend the Hursh and Silberberg (2008) framework and its future implementations. For example, alternatives might be derived from the BC family of transformations (e.g., BC and generalized BC, modulus). Although alternatives surely exist, it is most likely that the resulting transformations will ultimately reflect the same orderly distribution of values and will not further enhance the analysis. For instance, transformations using the log scale with Euler's number as a base will differ from transformation using the log scale with 10 as a base, but the orderly relationship between values remains the same. As such, the IHS transformation is likely just one of many transformations suited to evaluating operant demand.

Although the IHS scale presented here preserved the orderly distribution of observed consumption values, we must reiterate that the departure from the log scale has the potential to introduce new sources of variability. The IHS scale presented here is one that has been customized to emulate the $\log_{10}$ scale, and as such, is virtually identical to the $\log_{10}$ scale for values at 10 and above. However, it is the expected behavior of the IHS scale to differ from the $\log_{10}$ scale when values are below 10 and ultimately approach zero. Within this range, values arising from both the $\log_{10}$ and IHS scale remain orderly and related, but ultimately differ, and this will influence estimates of elasticity in the range of values nearing zero. For example, IHS and log transforms are within 2% for a data value of 5, and the difference is less than 10% down to a value of 3. Although these differences are small, the relationship between P_{MAX} and unit elasticity (derived via differentiation) is lost when one or both dimensions are below the index value (10) and P_{MAX} in IHS–IHS scale will not correspond with an elasticity value of -1. As such, differentiation can be used to solve for -1 but this solution may yield a value of proportional change that does not use constant units. Put simply, solutions here may approximate P_{MAX} but ultimately differ when demand is reconstituted in the natural scale. Given this variability, we recommend that users more simply evaluate the O_{MAX} in the natural scale using one of the tools provided here.

Conclusions

The IHS scale and the ZBE models presented here represent a novel implementation of contemporary methods used in operant demand. Novel implementations were necessary to fulfill the original goals of the Hursh and Silberberg (2008) framework. In particular, the ZBEn model with normalized α (Eq. 9) addresses several of the pragmatic and conceptual limitations of the original EXPL model. Specifically, it accommodates zero consumption data, eliminates the need for the k parameter (which complicates fitting and comparisons), and addresses the potential covariance of α with differences in span (k). Further, the ZBEn model appears to perform as well and, in some cases, better than existing models in a range of basic and applied cases. As such, we advocate for further exploration and validation of the ZBE model but caution researchers not to presume that any single model or combination will best characterize all instances of operant demand. That is, we do not advocate for researchers to abandon the EXPL or EXPD models altogether if those alternatives can answer research questions with fewer sources of variability and complexity. For example, in the absence of zero consumption values, it may be simpler and more pragmatic to use an earlier model (e.g., EXPL). In conclusion, we encourage researchers to consider the IHS scale and other log-like transformations that may extend the ZBE model.

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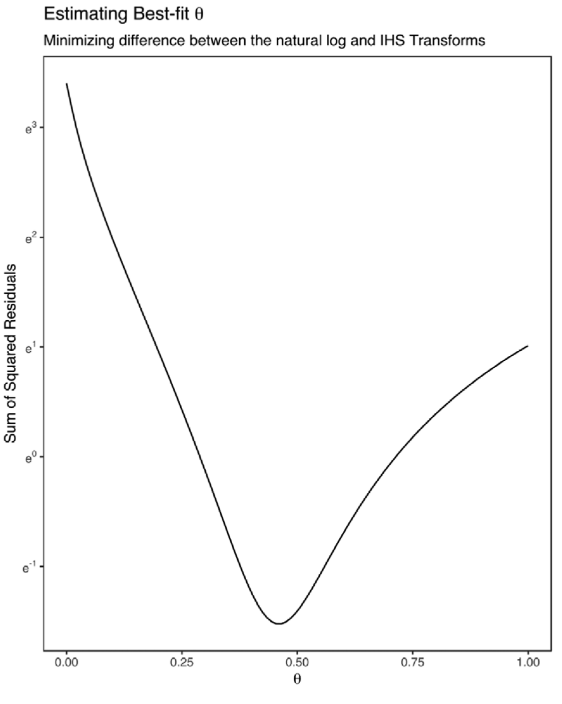
Appendix: Calculations for Mean Squared Percentage Error in the Exponential Model of Operant Demand

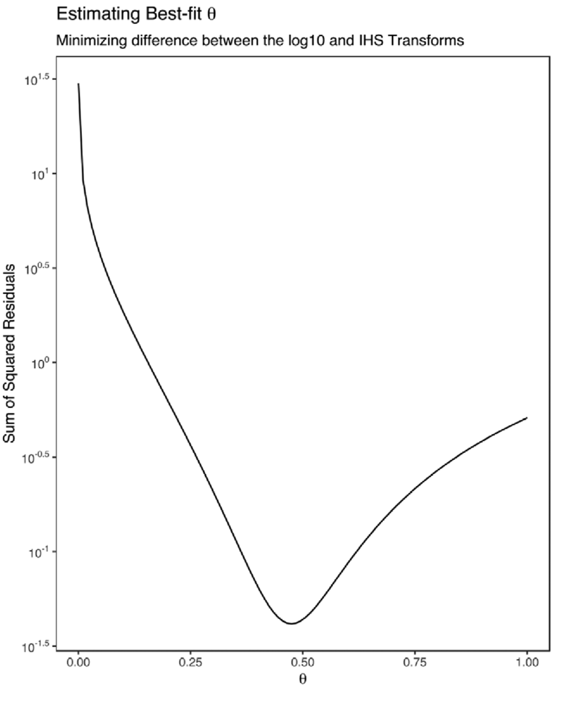
This section of the appendix illustrates how least squares regression performed in the log scale represents error as a proportion, rather than an absolute difference between a projected function and observed data. In this hypothetical demand series, a fitted function with a Q_0 of 1171, a K of 1.5, and an α of 0.00006 is applied to the following prices and observed consumption. As highlighted in the grey portions, the residual error on the log scale (Log_{10} Error) is the logarithm of the ratio of demand to consumption—itself a form of a percentage.

Price	Consumption	Log_{10} Consumption	Log_{10} Demand	Log_{10} Error	Linear Demand	$\frac{\text{Linear Demand}}{\text{Consumption}}$	$\text{Log}_{10}\left(\frac{\text{Linear Demand}}{\text{Consumption}}\right)$
0	1000	3	3.068556	0.068556	1171	1.171	0.068556
0.5	1000	3	3.016776	0.016776	1039.38570	1.039385	0.016776
1	1000	3	2.966784	-0.033215	926.369053	0.926369	-0.033215
1.5	800	2.90309	2.918517	0.015427	828.928538	1.036160	0.015427
2	800	2.90309	2.871916	-0.031173	744.588501	0.930735	-0.031173
2.5	700	2.845098	2.826924	-0.018173	671.311708	0.959016	-0.018173
3	600	2.778151	2.783485	0.005334	607.414633	1.012357	0.005334
4	500	2.69897	2.701054	0.002084	502.405167	1.004810	0.002084
5	400	2.60206	2.624215	0.022155	420.935719	1.052339	0.022156
10	200	2.30103	2.311500	0.010470	204.880527	1.024402	0.010471
15	100	2	2.091420	0.091420	123.429897	1.234298	0.091420

Determination of Parameter θ in IHS Transformation

The θ value used in this report was determined through a process of maximum





likelihood estimation designed to minimize the difference between the IHS and natural log and $\log_{10}$ transformations. That is, the IHS transformation was designed to simulate the natural log via θ and $\log_{10}$ via θ prior to casting the IHS into $\log_{10}$ units, $\ln(10)$. The figure below illustrates the best fit θ values of 0.4599 and 0.4987 for the natural log and $\log_{10}$ transformations, respectively. A difference exists here because the IHS transformation in $\log_{10}$ units essentially involves a sequence of two transformations. As a matter of convenience, we adopted a θ of 0.5 for use in subsequent formulae simulating the $\log_{10}$ transformation.

Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website.